Coupling and Weak Convergence

Richard Lockhart (Reg and Peter)

May 30, 2019

Today's Subjects

- Reg.
- Guttorp, Kulperger, and Lockhart. (1985) A coupling proof of weak convergence, *J Appl Prob*, **22**, 447-453
- Guttorp and Kulperger. (1984) Statistical inference for some Volterra population processes in a random environment. CJS, **13**, 289-302.
- Kulperger and Guttorp. (1981) Criticality conditions for some random environment population processes. Stoch Proc Appl, **11**, 207-212.
- Coupling
- Wasserstein metric, Hungarian construction, perfect sampling.

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- Guttorp, Kulperger, and Lockhart. (1985) A coupling proof of weak convergence, *J Appl Prob*, **22**, 447-453
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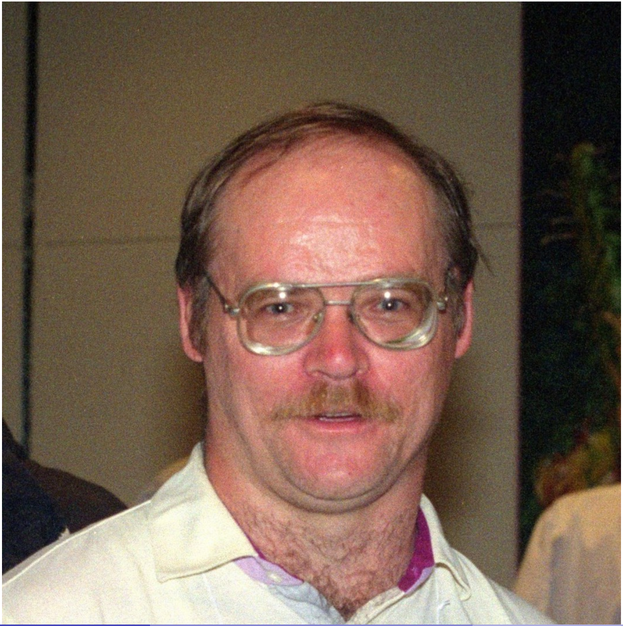
- Kulperger and Guttorp. (1981) Criticality conditions for some random environment population processes. Stoch Proc Appl, **11**, 207-212.
- Coupling
- Wasserstein metric, Hungarian construction, perfect sampling.
- We were children at the time.

Reg: 1992 Edmonton



Reg and Richard

Motivating problem

- Reg and Peter studying population processes.
- Stochastic version of Volterra differential equation models
- One example: population X_t driven by Brownian Motion B_t:

$$dX_t = X_t(\mu - \lambda X_t)dt + \sigma X_t dB_t$$

- Estimate μ, λ and σ based on continuous data on $[0, T]$.
- Take logs: $Z_t = \log X_t$ satisfies

$$dZ_t = \left(\mu - \sigma^2/2 - \lambda e^{Z_t}\right) dt + \sigma dB_t.$$

- I think I got involved over the case $\mu = \sigma^2/2$.

Coupling

- Slutsky: If $X_n \Rightarrow X$ and $d(Y_n, X_n) \to 0$ then $Y_n \Rightarrow X$.
- Random elements of a metric space of functions, say.
- Application requires X_n and Y_n to live in same probability space.
- Often X_n is an approximation to Y_n like a Taylor expansion.
- For us: know law of X_n and $X_n \Rightarrow X$.
- And Y_n is some process with similar dynamics to X_n.
- Coupling strategy: pick special probability space Ω.
- Construct X_n, Y_n on Ω so that $d(X_n, Y_n)$ small, probably.

Discrete Time example

- Y is random walk

$$P[Y(n+1) = k+1 \mid Y(n) = k] = \frac{1}{2} + r(k)$$
$$P[Y(n+1) = k-1 \mid Y(n) = k] = \frac{1}{2} - r(k).$$

- In application $r(k)$ small when k large.
- X is random walk with $r(0) = 1/2$ and $r(k) = 0$ for $k > 0$.
- X is absolute value of fair random walk so for $X_n(t) \equiv X(nt)/\sqrt{n}$

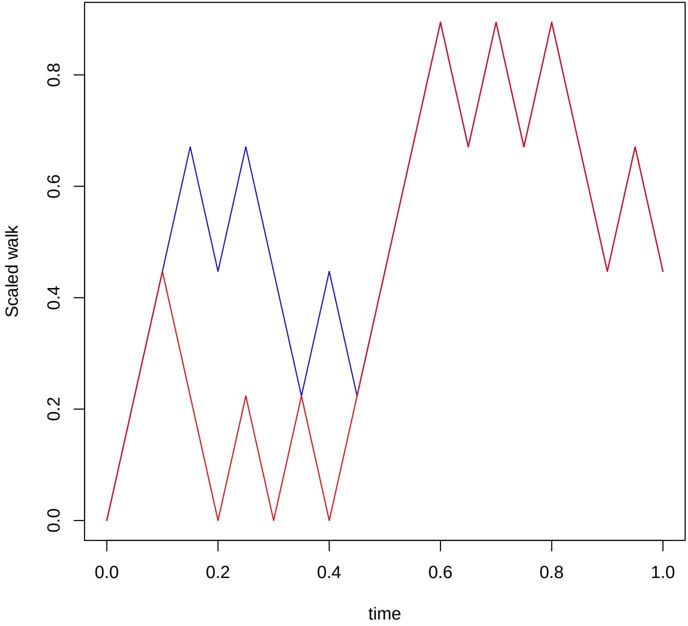
$$X_n \Rightarrow |B|.$$

- Same thing true for $Y_n(t) = Y(nt)/\sqrt{n}$ provided
  1. $\forall j(r(j) \geq 0)$;
  2. $\lim_{j\to\infty} jr(j) = 0$;
  3. $\inf\{r(j) : j < 0\} > 0$.

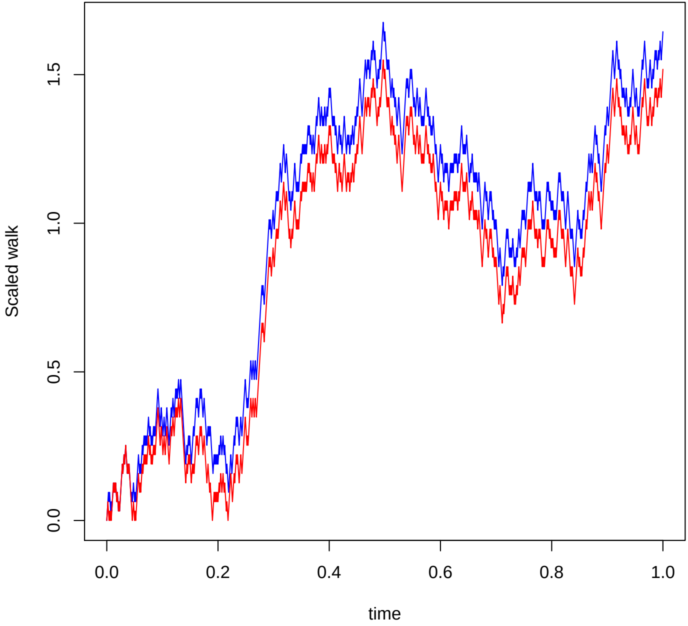
Easy strategy

- Construct X and Y so that they move in the same direction as often as possible.
- Use a single set $\{U_1, U_2, \cdots\}$ of iid Uniforms.
- Move $X(n)$ up if $U_n < 1/2$ (for $X(n) > 0$).
- Move $Y(n)$ up if $U_n < 1/2 + r(Y(n))$.

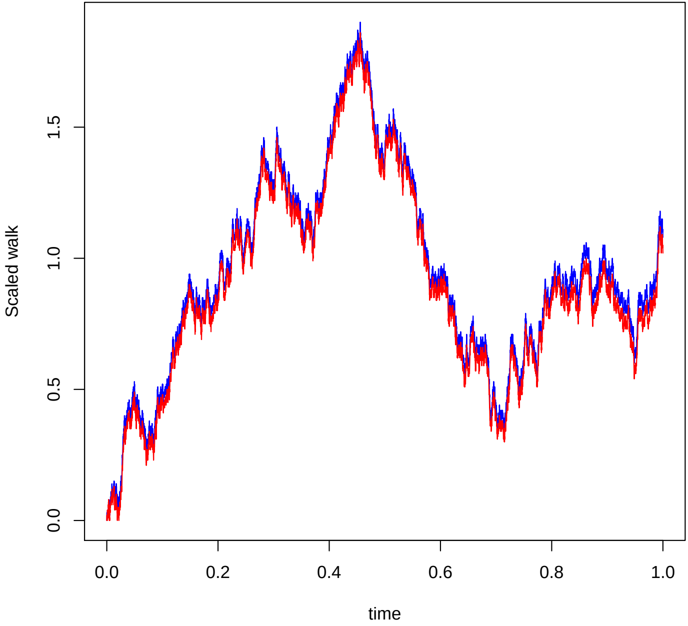
Simulation $n = 20$, $r = c/k^2$



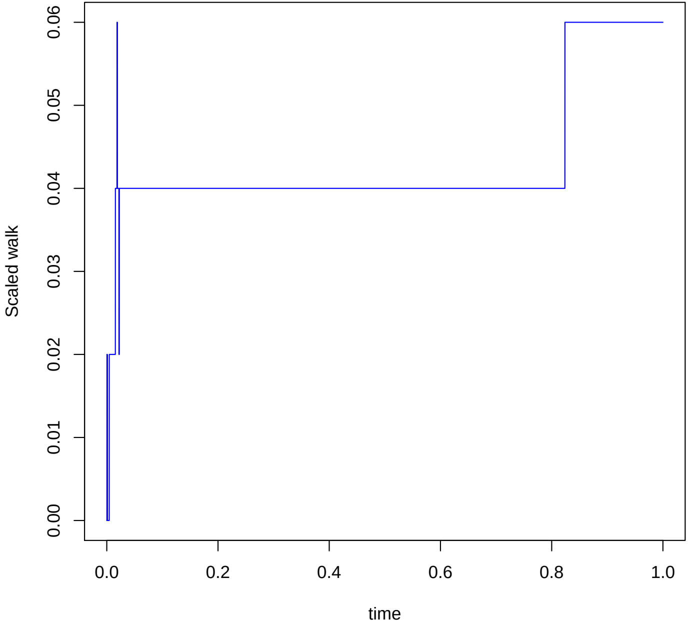
Simulation $n = 1000$



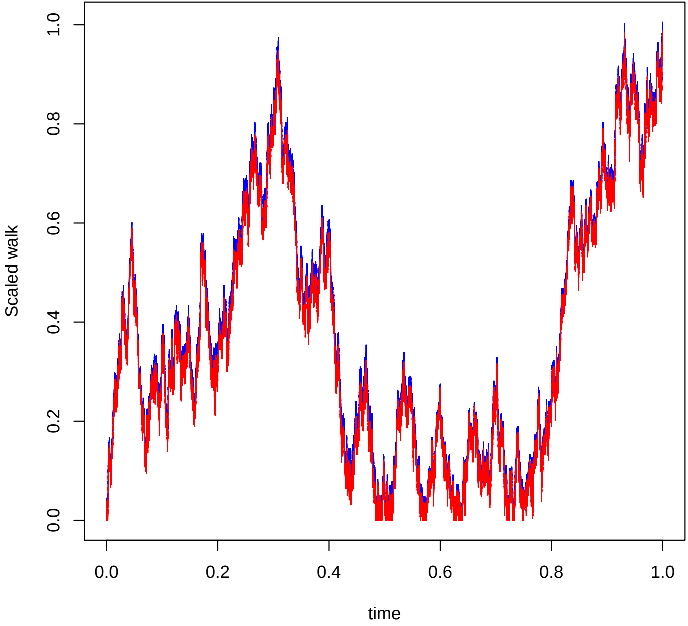
Simulation $n = 10000$



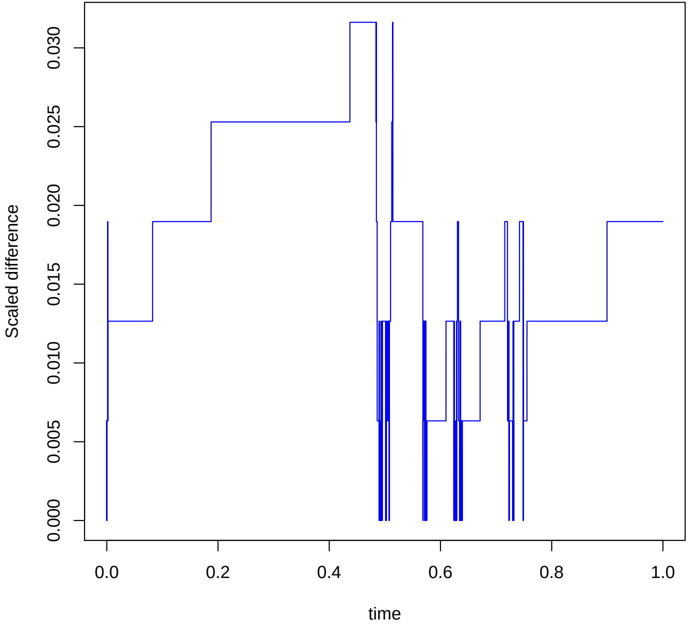
Difference $n = 10000$



Simulation $n = 100000$



Difference $n = 100000$



Reg Letter 1981 Nov

McMASTER UNIVERSITY
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24/11/81

Dear Richard

This is a follow up to last week. This is the random walk with typographical corrections.* I only went over the earlier version hurriedly so that an "approximate final" copy would get to you before 20/11/81.

Also when I spoke to you last month I forgot to ask if I left a silver fountain pen in Vancouver last summer.

Reg

* I make no guarantees on no more typographical or other errors – but I think it is okay now.

Nov 1981 Version

§1 Introduction.

We consider a random walk which takes jumps of 1 or −1, and which has a positive drift. We obtain conditions on the drift which yield a positive normal limit law. We first consider the random walks reflected at 0. The method used is to couple the process to an ordinary (no drift) random walk reflected at the origin, for which the positive normal limit law follows by the central limit theorem. We show the difference between the two processes is small at a good enough rate. Using Doob's decomposition (Chung (1974), p. 321) and then examining the increasing part was not useful here.

Coupling arguments, when they work, have a beauty of simplicity. The general idea is to associate with a process of interest, another process, or sequences of processes, which are simpler, and then if possible transfer over some property to the original process of interest. One example is in Hoel, Port and Stone (1972, p. 73). Another for a continuous time process defined by a stochastic differential equation is given in Kulperger and Guttorp (1981).

§2 A Reflected Random Walk with Drift

Notes from August 1981?

DIFFUSIONS

Problem: Constructing Solutions of

$$dZ_t = f(Z_t)\,dt + dB_t$$ where f is a given

monotone decreasing non-negative function. $f \geq 0$.

A solution is a process $\{Z_t : t \in [0,\infty)\}$

adapted to the natural B_t filtration which is

(i) continuous

(ii) strong Markov

(iii) satisfies $Z_t = \int_0^t f(Z_s)\,ds + B_t$ for all t

for almost all ω.

MoreNotes

Proof of Theorem:

An operator $S_x : C[0,\infty) \to C[0,\infty)$ is defined by

$$S_x y(t) = \int_0^t f(y(s))\,ds + x_t$$

Properties of S:

(i) $y_1 \le y_2 \Rightarrow S_x y_1 \ge S_x y_2$

(ii) S_x carries bounded mble ftns y to $C[0,\infty)$.

(iii) S_x is jointly mble in x and y.

We define a family L_α, U_α of maps indexed by countable ordinals α by transfinite induction

$$L_0 x = x \; ; \; U_0 x = S_x x$$

$$L_\alpha x = S_x\left(\inf_{\beta<\alpha} U_\beta x\right); \; U_\alpha(x) = S_x L_\alpha x$$

March 1982 Version

SUMMARY

A diffusion X_t with non-negative drift $h(x)$ and variance 1 is coupled to a reflected Brownian motion. The coupling is used to find conditions under which $t^{-1/2} X_t$ has a half-normal limit law as $t \to \infty$.

Accepted 1985 Version

Abstract

Weak convergence to reflected Brownian motion is deduced for certain upwardly drifting random walks by coupling them to a simple reflected random walk. The argument is quite elementary, and also gives the right conditions on the drift. A similar argument works for a corresponding continuous-time problem.

RANDOM WALK; DIFFUSIONS WITH DRIFT

1. Introduction

Guttorp and Kulperger (1984) studied estimation problems for some Volterra-type population processes in a random environment. In one situation, attention was focused on the behavior of a process with a steady upwards drift, diminishing as the process moved far above the origin. A discrete approximation is a random walk with drift. In this note we study the weak convergence of certain Harris random walks to reflected Brownian motion, finding sufficient and essentially necessary conditions for the convergence. The proofs are obtained by coupling, i.e. by constructing the process of interest on the same probability space as a reflected fair random walk, and bounding the difference between the processes.

Other kinds of coupling

- Couple two Markov Chains X_n, Y_n together with X_n stationary and both with same stationary transitions.
- Run independently till they collide; then couple together to show Y_n is asymptotically stationary.
- Perfect sampling / coupling from the past.
- Use Uniform U_{-1} to determine transitions from X_{-1} to X_0 and U_{-2} for X_{-2} to X_{-1}.
- Keep U_i fixed and go back in time far enough that every starting point leads to same value of X_0.
- The Hungarian construction: on a single probability space construct Brownian Bridge and emprical process so two are close together.
- Used to measure distance between distributions; derive goodness-of-fit tests.

Take away messages

- Research is slow.

Take away messages

- Research is slow.
- And fun with friends.

Take away messages

- Research is slow.
- And fun with friends.
- E-mail is helpful.

Take away messages

- Research is slow.
- And fun with friends.
- E-mail is helpful.
- I am grateful to Reg for years of friendship.